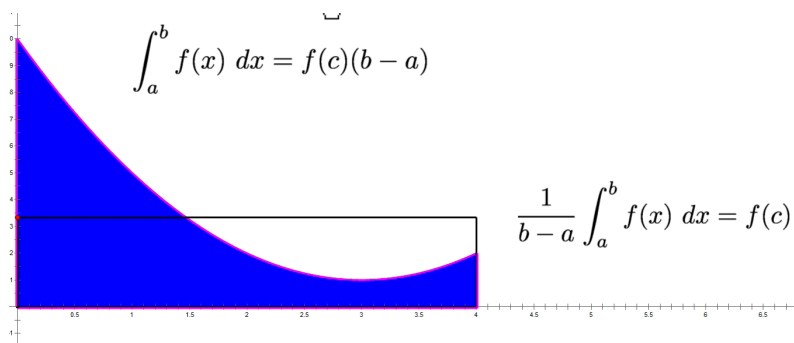


4.4 The MVT for integrals (average value over an interval) and the 1st & 2nd FTC

Chris Thiel, OFMCap, 2021

Average Value on a Interval

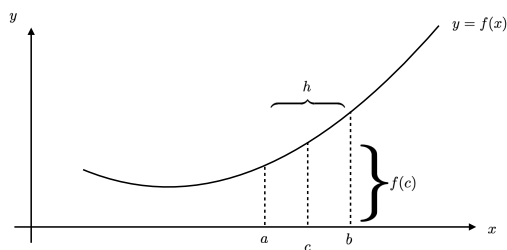
Mean Value Theorem for Integrals



(Thanks Jim O'Connor for this Animation!)

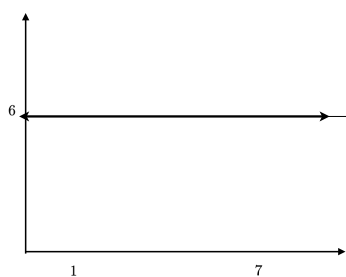
There must be a c in $[a, b]$ where

$$f(c) \cdot (b - a) = \int_a^b f(x) \, dx$$



$f(c)$ is the average value of f on the interval $[a, b]$

$$f(c) = \frac{1}{b-a} \int_a^b f(x) \, dx$$



Using Geometry..

$$\int_1^7 f(x) \, dx = (7-1)(6) = 36$$

$$\frac{1}{7-1} \int_1^7 f(x) \, dx = \frac{36}{6} = 6$$

Using Calculus..

$$\int f(x) \, dx = F(x) = 6x + C$$

If $F(0) = 0$ then $C = 0$

$$F(x) = 6x \text{ and } F'(x) = f(x) = 6$$

Notice this..

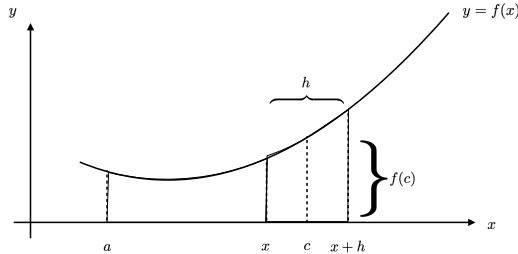
$$F(7) = 6(7) = 42$$

$$F(1) = 6(1) = 6$$

$$42 - 6 = 36 = \int_1^7 f(x) \, dx = F(7) - F(1) \quad \text{Will this ALWAYS work?}$$

Claim

$$\int_a^b f(x) dx = F(b) - F(a)$$



$$A(x) = \int_a^x f(t) dt \quad \dots \text{and} \dots \quad \int f(x) dx = F(x) + C, \text{ if } F'(x) = f(x)$$

$$A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \rightarrow 0} \frac{f(c) \cdot h}{h} = \lim_{h \rightarrow 0} f(c) = f(x) \quad \text{so } A(x) \text{ is an antiderivative of } f(x)$$

$$A(x) = F(x) + C = \int_a^x f(t) dt \quad \text{consider } A(a) = \int_a^a f(t) dt = 0 = F(a) + C$$

$$\text{Here } x = a \text{ so } F(a) + C = 0$$

$$\text{Then } \int_a^x f(t) dt = F(x) - F(a)$$

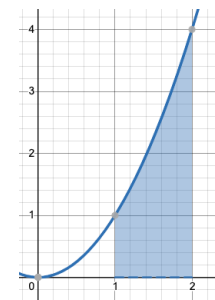
$$\dots \text{ so } C = -F(a)$$

$$\text{If we let } x = b \text{ we have the FTC: } \int_a^b f(x) dx = F(b) - F(a)$$

4.4 The First Fundamental Theorem of Calculus

$$\int_a^b f(x) dx = F(b) - F(a)$$

$$F(a) + \int_a^b f(x) dx = F(b)$$



$$\int_1^2 x^2 dx \quad \text{So } F(x) = \frac{x^3}{3}$$

$$= \frac{x^3}{3} \Big|_1^2 = F(2) - F(1)$$

$$= \frac{2^3}{3} - \frac{1^3}{3}$$

$$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

The Net Change "Theorem"

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

$$F(a) + \int_a^b f(x) \, dx = F(b)$$

A tank contains 10 gallons. Water is added at a rate of 4 gallons per minute, but leaks at $\sqrt{t}$ gallons per minute for time $t \geq 0$. How much is in the tank after 30 minutes?

$$10 + \int_0^{30} 4 - t^{1/2} \, dt$$

$$10 + 4t - \frac{2}{3}t^{3/2} \Big|_0^{30}$$

$$10 + 4(30) - \frac{2}{3}30^{3/2} \Big|_0^{30}$$

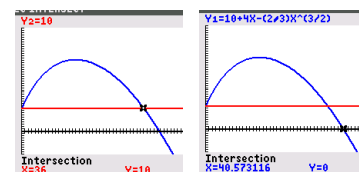
$$\approx 20.455 \text{ gallons}$$

How long will it take to have 10 gallons again?
How long will it take to be empty?

$$Y_1 = 10 + \left(4(t) - \frac{2}{3}t^{3/2}\right)$$

$$Y_2 = 10$$

$$Y_3 = 0$$



The tank will have 10 gallons again after 36 minutes and be empty in 45.573 minutes.

The Second Fundamental Theorem of Calculus

$$\frac{d}{dx} \left[\int f(x) \, dx \right] = f(x)$$

Or the "Chain Rule" Version:

$$\frac{d}{dx} \left[\int_a^{g(x)} f(t) \, dt \right] = f(g(x)) \cdot g'(x)$$

$$\frac{d}{dx} \int_2^x \tan(t^3) \, dt = \tan x^3$$

$$\frac{d}{dx} \int_{-1}^{x^3} \frac{1}{1+t} \, dt = \frac{1}{1+x^3} \cdot 3x^2 = \frac{3x^2}{1+x^3}$$

$$\frac{d}{dx} \int_2^{\sin x} \sqrt[3]{1+t^2} \, dt = \sqrt[3]{1+\sin^2 x} \cdot \cos x$$

$$\frac{d}{dy} \int_{\pi}^{3y} 14x^2 \, dx = 14(3y)^2 \cdot 3 = 378y^2$$